A SIC in the rough

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1 Introduction

A POVM is the most general form of a quantum measurement. Unlike projective measurements, POVMs can be *informationally complete*: quantum states can be completely represented as probability distributions with respect to such measurements. Today we consider the *manifold* formed by a particular class of POVMs. But to have a manifold we need some way of measuring distances. QBists often like to consider a quantity which quantifies how the Born rule with respect to a reference measurement diverges from the law of total probability. We'll use this to define a notion of distance on a quotient of the POVM manifold.

2 A triangle of POVMs

We'll keep things simple. Suppose we have a rank-1, unbiased, minimal informationally complete POVM on a qubit. Recall that a POVM with a finite number of outcomes corresponds to a set of positive semidefinite matrices which sum to the identity,

$$E_i \geq 0 \qquad\qquad \sum_{i=1}^{n} E_i = I. \tag{2.1}$$

Unbiased means that $\forall i : \mathrm{tr}(E_i) = \text{const}$. Taking the trace of the summing-to-the-identity condition, we have $n\mathrm{tr}(E_i) = d$, so $\mathrm{tr}(E_i) = d/n$. Using this and the rank-1 condition means that we can write any POVM element as

$$E_i = \frac{d}{n}|\psi_i\rangle\langle\psi_i|, \tag{2.2}$$

where $\{|\psi_i\rangle\}$ is a set of pure states. For a POVM to be minimal informationally complete, it must have d^2 elements which are linearly independent: so we'll take $n = 4$. Next, remember that we can write any pure qubit state as a linear combination of Pauli matrices,

$$|\psi_i\rangle\langle\psi_i| = \frac{1}{2}\Big(I + x_i\sigma_x + y_i\sigma_y + z_i\sigma_z\Big), \tag{2.3}$$

where $x^2 + y^2 + z^2 = 1$. Letting $\mathbf{n}_i = (x, y, z)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ we have

$$E_i = \frac{1}{4}\Big(I + \mathbf{n}_i \cdot \vec{\sigma}\Big). \tag{2.4}$$

The rank-1 condition translates into $\|\mathbf{n}_i\| = 1$ and from

$$\sum_i E_i = I + \frac{1}{4}\sum_i \mathbf{n}_i \cdot \vec{\sigma} = I, \tag{2.5}$$

we conclude that $\sum_i \mathbf{n}_i = 0$. Putting this all together means that the POVM can be represented as a set of 4 points on the unit sphere which average to center of the sphere. Informationally completeness means that these *Bloch vectors* are linearly independent.

Let $G_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j$ be the Gram matrix of $\{\mathbf{n}_i\}$. This will be a 4×4 postive semidefinite symmetric matrix. Moreover, consider that $\|\mathbf{n}_i\| = 1$ and $\sum_i \mathbf{n}_i = 0$ put further restrictions of G. First, its diagonal must be all 1's. But also $\sum_i G_{ij} = \sum_i \mathbf{n}_i \cdot \mathbf{n}_j = 0$: the sum of columns (and rows) must be 0. A first attempt:

$$G(a,b) = \begin{pmatrix} 1 & a & b & -1-a-b \\ a & 1 & c & -1-a-c \\ b & c & 1 & -1-b-c \\ -1-a-b & -1-a-c & -1-b-c & 1 \end{pmatrix}. \tag{2.6}$$

But actually summing the last column (or row) gives us a relationship between a, b, c: $a + b + c = -1$. We conclude that G is a function of just two parameters $a, b \in [0, 1]$,

$$G = \begin{pmatrix} 1 & a & b & -1-a-b \\ a & 1 & -1-a-b & b \\ b & -1-a-b & 1 & a \\ -1-a-b & b & a & 1 \end{pmatrix}. \tag{2.7}$$

In fact, the vectors $\{\mathbf{n}_i\}$ can be recovered from the Gram matrix up to overall rotations. We conclude that the space of unbiased rank-1 qubit POVMs with 4 outcomes, quotiented by overall 3D rotations, is two dimensional. This is great news for us since we want to visualize it.

It turns out that G can be diagonalized by a Hadamard matrix. Its eigenvalues are

$$\lambda = \Big\{0, 2(a+1), 2(b+1), -2(a+b)\Big\}, \tag{2.8}$$

and letting

$$H = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{2.9}$$

we have

$$G = H D_\lambda H^T, \tag{2.10}$$

where D_λ is the diagonal matrix of the eigenvalues. The columns of H are the eigenvectors of G. The condition for G to be positive semidefinite, and so a valid Gram matrix, is for its eigenvalues to be all nonnegative. Now G has at most rank 3 since the vectors live in a three dimensional space: for the POVM to be informationally complete, we require the rank of G to be exactly three. Thus let

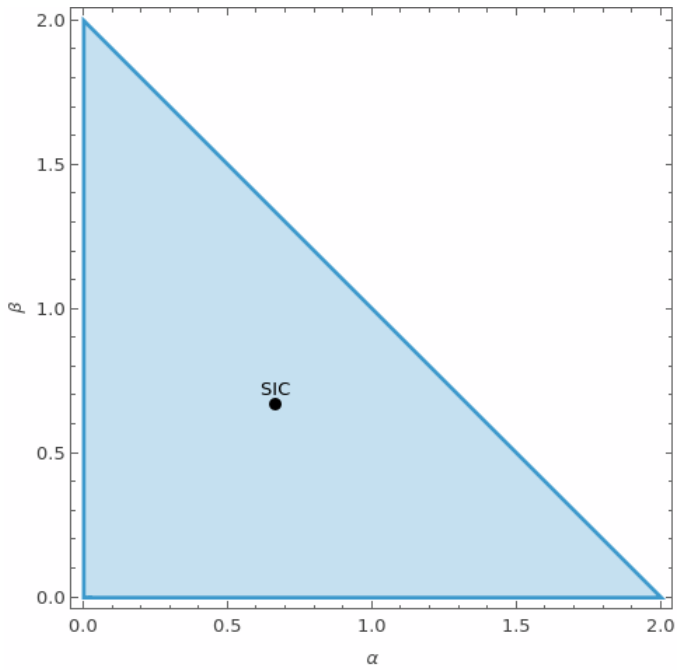
$$\alpha = a + 1 > 0 \qquad \beta = b + 1 > 0 \qquad \gamma = -a - b > 0, \tag{2.11}$$

where $\alpha + \beta + \gamma = 2$. In terms of these variables, the space of POVMs is

$$\alpha, \beta > 0 \qquad \alpha + \beta < 2, \tag{2.12}$$

which is the interior of a triangle!

Let's explore this geometry. Each point in the triangle corresponds to a tetrahedron up to rotations. Suppose $\alpha = \beta = \gamma$. This implies $\alpha = \beta = \gamma = 2/3$ and $a = b = -1/3$. In this case, $-1 - a - b = -1/3$ so that the off diagonal entries of G are $-1/3$: the vectors all make equal angles with each other. This is the defining condition of a SIC-POVM, which forms a regular tetrahedron in the Bloch sphere. For us, it corresponds to the centroid of our triangle, the average of the vertices $(0, 0), (0, 2)$ and $(2, 0)$.



As you move closer to the edges of the triangle, the eigenvalues shrink until on the edges themselves we lose informational completeness. Using the eigendecomposition we can define $V = H\sqrt{D_\lambda}$ so that $VV^T = G$. In fact, since one of the eigenvalues is 0, we can drop the first column and take the four vectors to be the rows of

$$V(\alpha, \beta) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{\alpha} & -\sqrt{\beta} & \sqrt{\gamma} \\ -\sqrt{\alpha} & \sqrt{\beta} & -\sqrt{\gamma} \\ \sqrt{\alpha} & -\sqrt{\beta} & -\sqrt{\gamma} \\ \sqrt{\alpha} & \sqrt{\beta} & \sqrt{\gamma} \end{pmatrix}. \tag{2.13}$$

The four rows of this matrix give the vertices of the corresponding tetrahedron.

3 A QBist metric

Now let's think about how to define a notion of distance between these tetrahedra. To get there, let us discuss how the Born rule, the rule for calculating probabilities in quantum mechanics, appears when one represents quantum states and effects with respect to a reference measurement. Because the measurement is informationally complete the probabilities $P(E_i|\rho) = \mathrm{tr}(E_i\rho)$ suffice to characterize the quantum state ρ. We imagine that upon getting outcome E_i of the measurement, the state $\sigma_i = E_i/\mathrm{tr}(E_i)$ is prepared. Then any effect A can be characterized by the conditional probabilities $P(A|E_i) = \mathrm{tr}(A\sigma_i)$: conditional on getting outcome E_i on the reference, what is the probability of getting outcome A in a subsequent measurement? In particular, the conditional probability matrix $P(E_i|E_j) = \mathrm{tr}(E_i\sigma_j)$ characterizes the reference measurement itself, the reference states in terms of the reference effects and vice versa. For our qubit reference measurement, we have

$$P_{ij} \equiv P(E_i|E_j) = \mathrm{tr}(E_i\sigma_j) = \frac{1}{2}\mathrm{tr}(\sigma_i\sigma_j) = \frac{1}{4}(1 + \mathbf{n}_i \cdot \mathbf{n}_j), \tag{3.1}$$

since $\sigma_i = \frac{1}{2}(I + \mathbf{n}_i \cdot \vec{\sigma})$ and the Pauli matrices form an orthogonal operator basis. In other words,

$$P = \frac{1}{4}(J + G). \tag{3.2}$$

Now since $\{\sigma_i\}$ themselves form a linearly independent basis, we can write $\rho = \sum_j w_j \sigma_j$. But then

$$P(E_i|\rho) = \mathrm{tr}(E_i\rho) = \sum_j w_j \mathrm{tr}(E_i\sigma_j) = \sum_j P(E_i|E_j)P(E_j|\rho), \tag{3.3}$$

so that in vector notation $P(E|\rho) = P(E|E)P(E|\rho)$. Moreover, because of informational completeness, $P(E|E)$ will be invertible. Let $\Phi = P^{-1}$, so that $w = \Phi P(E|\rho)$. For any effect A then,

$$P(A|\rho) = \mathrm{tr}(A\rho) = \sum_{ij} \Phi_{ij} P(E_j|\rho)\mathrm{tr}(A\sigma_i) = \sum_{ij} P(A|E_i)\Phi_{ij}P(E_j|\rho), \tag{3.4}$$

or more compactly, $P(A|\rho) = P(A|E)\Phi P(E|\rho)$. We see that the Born rule has become a gentle modification of the law of total probability. Indeed, if Φ were the identity, then the Born rule were reduce to the LTP: but this cannot be achieved for any measurement in quantum theory. It is known, however, that among MIC measurements, the closest one can get is $\Phi = (d+1)I - (1/d)J$ so that

$$P(A|\rho) = \sum_i P(A|E_i)\left\{(d+1)P(E_i|\rho) - \frac{1}{d}\right\}. \tag{3.5}$$

This is achieved precisely by a SIC-POVM. In particular, among all MIC's (and actually among all choices of post-measurement states $\{\sigma_i\}$), SIC-POVMs minimize $\|I - \Phi\|$ with respect to any unitarily invariant matrix norm as Blake Stacey, John DeBrota, and Chris Fuchs showed long ago.

With this in mind, let us define a measure of distance between two POVMs E and F (up to overall rotations),

$$D(E,F) = \|\Phi_E - \Phi_F\|, \tag{3.6}$$

where we stick to the Frobenius norm for simplicity. According to this measure, two POVMs are close precisely when their modifications of the law of total probability are similar. And as we've seen, we can think of $D(E,F)$ as defining a distance measure on our triangle. Let's see if we can get a nice expression for it. We've already seen that $G = HD_\lambda H^T$ where $D_\lambda = \mathrm{diag}[(0, 2\alpha, 2\beta, 2\gamma)]$ for every POVM in the triangle. So

$$P = \frac{1}{4}(J + HD_\lambda H^T). \tag{3.7}$$

But notice that the first column of H is $(1,1,1,1)^T/2$. If we let $D_1 = \mathrm{diag}[(1,0,0,0)]$, we have $HD_1H^T = (1/4)J$ so that

$$P = \frac{1}{4}(4HD_1H^T + HD_\lambda H^T) = H\begin{pmatrix} 1 & & & \\ & \alpha/2 & & \\ & & \beta/2 & \\ & & & \gamma/2 \end{pmatrix} H^T, \tag{3.8}$$

and

$$\Phi = P^{-1} = H\begin{pmatrix} 1 & & & \\ & 2/\alpha & & \\ & & 2/\beta & \\ & & & 2/\gamma \end{pmatrix} H^T, \tag{3.9}$$

and consequently,

$$D^2(E,F) = \|\Phi_E - \Phi_F\|^2 = \|HD_EH^T - HD_FH^T\|^2 \tag{3.10}$$

$$= \|H(D_E - D_F)H^T\|^2 = \|D_E - D_F\|^2 \tag{3.11}$$

$$= \left(\frac{2}{\alpha_E} - \frac{2}{\alpha_F}\right)^2 + \left(\frac{2}{\beta_E} - \frac{2}{\beta_F}\right)^2 + \left(\frac{2}{\gamma_E} - \frac{2}{\gamma_F}\right)^2, \tag{3.12}$$

where we've used the unitary invariance of the Frobenius norm. If we choose coordinates

$$X = \frac{2}{\alpha} \qquad Y = \frac{2}{\beta} \qquad Z = \frac{2}{\gamma}, \tag{3.13}$$

then we have

$$D^2(E,F) = (X_E - X_F)^2 + (Y_E - Y_F)^2 + (Z_E - Z_F)^2, \tag{3.14}$$

which is just the usual Euclidean distance on $\mathbb{R}^3$! Translating the constraints that $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma = 2$ into these coordinates shows us that we can view our POVM manifold as the set

$$\left\{(X,Y,Z) > 0 : \frac{1}{X} + \frac{1}{Y} + \frac{1}{Z} = 1\right\}. \tag{3.15}$$

In other words, our choice of distance function on the triangle turns it into a curved manifold, which we can embded isometrically into three dimensional Euclidean space as the surface statisfying these constraints.

To get a visceral sense of what this means, check out this visualizer whipped up lightning fast by our friend ChatGPT. Drag the blue point around the flat triangle and see how it translates into a point on the curved surface, as well as the corresponding tetrahedron representing Bloch vectors of the POVM. Notice that as one approaches the edges of the triangle, one zips along one of the "arms" of the surface to infinity; as one approaches a vertex of the triangle, one zips to infinity along the flatter regions between the arms. Indeed, dragging the point at a constant rate in the triangle leads to the point on the curved surface fleeing off to infinity ever faster. Conversely, moving at a constant velocity on the curved surface out to infinity means you move ever slower in the triangle: thus one only asymptotically approaches the boundaries of the triangle, where informational completeness disintegrates. Finally, the SIC sits at the point of greatest curvature, in the middle of the whole thing.

4 A little differential geometry

Finally, let's use this opportunity to practice our differential geometry and describe how our Φ-distnace function induces a metric on the triangle so that we can view it as an intrisically curved manifold. Consider the infinitesimal distance between a POVM E and $E + dE$,

$$ds^2 = D^2(E, E+dE) = \|\Phi_E - \Phi_{E+dE}\|^2 = \|d\Phi\|^2 = \mathrm{tr}(d\Phi^T d\Phi) \tag{4.1}$$

where $d\Phi = \Phi_E - \Phi_{E+dE}$. Using the result from above, we have

$$ds^2 = dX^2 + dY^2 + dZ^2 \tag{4.2}$$

$$= \left(d\left[\frac{2}{\alpha}\right]\right)^2 + \left(d\left[\frac{2}{\beta}\right]\right)^2 + \left(d\left[\frac{2}{\gamma}\right]\right)^2. \tag{4.3}$$

Since

$$\frac{d}{d\alpha}\frac{2}{\alpha} = -\frac{2}{\alpha^2}, \tag{4.4}$$

we have

$$ds^2 = 4\left(\frac{d\alpha^2}{\alpha^4} + \frac{d\beta^2}{\beta^4} + \frac{d\gamma^2}{\gamma^4}\right). \tag{4.5}$$

But since $\gamma = 2 - \alpha - \beta$ and $d\gamma = \frac{\partial\gamma}{\partial\alpha}d\alpha + \frac{\partial\gamma}{\partial\beta}d\beta$, we have

$$\frac{\partial\gamma}{\partial\alpha} = -1 \qquad \frac{\partial\gamma}{\partial\beta} = -1 \qquad d\gamma = -d\alpha - d\beta. \tag{4.6}$$

Therefore

$$ds^2 = 4\left(\frac{d\alpha^2}{\alpha^4} + \frac{d\beta^2}{\beta^4} + \frac{d\alpha^2 + 2d\alpha d\beta + d\beta^2}{\gamma^4}\right) \tag{4.7}$$

$$= \left[\frac{4}{\alpha^4} + \frac{4}{\gamma^4}\right] d\alpha^2 + \left[\frac{8}{\gamma^4}\right] d\alpha d\beta + \left[\frac{4}{\beta^4} + \frac{4}{\gamma^4}\right] d\beta^2. \tag{4.8}$$

It is convention to write a 2D metric as

$$ds^2 = \begin{pmatrix} d\alpha & d\beta \end{pmatrix} g(\alpha,\beta) \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} = g_{\alpha\alpha} d\alpha^2 + 2g_{\alpha\beta} dx dy + g_{\beta\beta} d\beta^2, \tag{4.9}$$

where

$$g(\alpha,\beta) = 4\begin{pmatrix} \alpha^{-4} + \gamma^{-4} & \gamma^{-4} \\ \gamma^{-4} & \beta^{-4} + \gamma^{-4} \end{pmatrix}. \tag{4.10}$$

So on the one hand, we can embed the manifold $\mathcal{M}$ into 3D Euclidean space using the map $\mathcal{E} : \mathcal{M} \to \mathbb{R}^3$ defined by

$$\mathcal{E}(\alpha,\beta) = 2\begin{pmatrix} \alpha^{-1} \\ \beta^{-1} \\ \gamma^{-1} \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{4.11}$$

where $\gamma = 2 - \alpha - \beta$. Equivalently, the "pullback" of the Euclidean metric in $\mathbb{R}^3$, that is, $\mathcal{E}^*(dX, dY, dZ)$, to the manifold itself is $g(\alpha,\beta)$. For example, at the SIC, $\alpha = \beta = 2/3$, so

$$g_{\text{SIC}} = \begin{pmatrix} 81/2 & 81/4 \\ 81/4 & 81/2 \end{pmatrix}. \tag{4.12}$$

From here, one can do all sorts of interesting differential geometry. We'll confine ourselves to considering the Gaussian curvature, which works out to be

$$K(\alpha,\beta,\gamma) = \frac{2\alpha^3\beta^3\gamma^3}{(\alpha^4 + \beta^4 + \gamma^4)^2}. \tag{4.13}$$

In two dimensions, in fact, the Gaussian curvature determines the entire Riemann curvature tensor. Clearly, $K > 0$ everywhere in the interior of the triangle: we have positive curvature, so geodesics tend to converge, circles have smaller circumferences than in the Euclidean case, and angles of triangles sum to more than 180°. At the SIC, we have $K = 4/27$, which is its maximum: initially parallel geodesics experience their strongest focusing near the SIC. And finally, as any coordinate approaches zero, $K \to 0$: the manifold becomes asymptotically flat. But we already knew all this from looking at the pictures!